Belief Uncertain Sets with an Absolute-Truth Marker: Mathematical Representations of Truth and Absolute Truth

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Abstract

Neutrosophic sets refine degree-based modeling by splitting information into three components: truth, indeterminacy, and falsity. This triplet representation is particularly useful when evidence is incomplete or inconsistent and one needs to capture, in a systematic way, situations that are "not fully true" and simultaneously "not fully false" [1–4]. Building on this viewpoint, Smarandache and other researchers have investigated neutrosophic structures equipped with an absolute-truth marker, which allows one to distinguish ordinary truth from an explicitly labeled absolute truth. In addition, by introducing a separate reliability or belief parameter, one can encode how strongly an assessment is trusted and thereby express absolute truth as the combination of maximal truth and maximal belief. In this paper, we systematize these ideas and study mathematically coherent ways to represent both Truth and Absolutely Truth within degree-based and neutrosophic-style frameworks.

Keywords: Belief Uncertain Sets, Uncertain Sets, Belief Sets, Absolute-truth marker

sectionContents of this paper The remainder of the paper is organized as follows.

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1 Preliminaries

This section fixes notation and recalls the basic notions used throughout the paper. Unless stated otherwise, all sets considered here are finite.

1.1 Fuzzy Sets and Neutrosophic Sets

In Zadeh's framework, vagueness is modeled by allowing partial membership: each element is assigned a degree in the unit interval rather than being declared simply in or out [5, 6]. Many well-known formalisms extend this idea by enriching the degree information, including Intuitionistic Fuzzy Sets [7], SuperHyperFuzzy Sets [8], and Hesitant Fuzzy Sets [9]. For completeness, we recall the standard definitions.

Definition 1.1 (Fuzzy Set)**.** [5] Let Y be a nonempty universe. A *fuzzy set* on Y is a function $\tau : Y \to [0,1]$. A *fuzzy relation* on Y is a fuzzy set on $Y \times Y$, i.e., a map $\delta : Y \times Y \to [0,1]$. We say that δ is a *fuzzy relation on* τ if, for all $y, z \in Y$,

$$\delta(y,z) \;\leq\; \min\{\tau(y),\, \tau(z)\}.$$

Neutrosophic sets further refine degree-based modeling by separating three aspects of information: truth, indeterminacy, and falsity. This tri-component description is useful when evidence is incomplete or conflicting and one wishes to represent "not fully true" and "not fully false" in a controlled way [1–4]. Related frameworks that broaden the neutrosophic viewpoint include double-valued neutrosophic sets [10, 11], bipolar neutrosophic sets [12, 13], and plithogenic sets [14, 15]. We use the single-valued version below.

Definition 1.2 (Single-valued Neutrosophic Set)**.** [1, 16] Let X be a nonempty set. A *single-valued neutrosophic set (SNS)* A on X consists of three functions

$$T_A : X \to [0,1], \qquad I_A : X \to [0,1], \qquad F_A : X \to [0,1],$$

where $T_A(x)$, $I_A(x)$, and $F_A(x)$ denote, respectively, the truth, indeterminacy, and falsity degrees of x with respect to A. These values satisfy, for every $x \in X$,

$$0 \;\leq\; T_A(x) + I_A(x) + F_A(x) \;\leq\; 3.$$

1.2 Uncertain Sets

An *Uncertain Set* provides a uniform template for assigning, to each element of a universe, an uncertainty value whose admissible range is fixed in advance by a chosen *degree-domain*. By varying the degree-domain, one can treat fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and related degree-based models as instances of a single scheme [17, 18].

Definition 1.3 (Uncertain Set (U-Set))**.** [17] Let X be a nonempty set and let M be an uncertainty model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$. A *U-Set of type* M on X is a mapping

$$\mu_M : X \longrightarrow \mathrm{Dom}(M).$$

For each $x \in X$, the value $\mu_M(x) \in \mathrm{Dom}(M)$ is called the *M-membership degree* (or *M-uncertainty value*) of x. Appropriate choices of M recover the standard degree-based set models (e.g., fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and others).

1.3 Neutrosophic set with an absolute-truth marker

A neutrosophic set with an absolute-truth marker assigns truth, indeterminacy, and falsity degrees, where truth may take a special value indicating universal absolute truth.

Definition 1.4 (Neutrosophic set with an absolute-truth marker)**.** Let X be a universe and fix $\varepsilon > 0$. A (single-valued) neutrosophic set on X is a triple of maps

$$A = (T_A, I_A, F_A), \qquad T_A, I_A, F_A : X \to]^{-}0, 1^{+}[.$$

We say that, for $x \in X$:

1. x is *(relatively) true in* A if $T_A(x) = 1$.
2. x is *absolutely true in* A if $T_A(x) = 1^+ = 1 + \varepsilon$.

(Optionally, one may also require $I_A(x) = 0$ and $F_A(x) = 0$ in these two cases to express "pure" truth without indeterminacy or falsity.)

The comparison of single-valued neutrosophic sets and neutrosophic sets with an absolute-truth marker is provided in Table 1.

Item	**Single-valued Neutrosophic Set (SNS)**	**Neutrosophic Set with an absolute-truth marker**
Underlying maps	$T, I, F : X \to [0, 1]$.	$T, I, F : X \to]^-0, 1^+[$ (with $1^+ = 1 + \varepsilon$).
Truth scale	Truth is bounded by 1 (maximum truth).	Truth may exceed 1 via the distinguished value 1^+ to mark absolute truth.
Meaning of "true"	(Fully) true typically means $T(x) = 1$.	Relatively true means $T(x) = 1$; absolutely true means $T(x) = 1^+$.
Constraint	$0 \leq T(x) + I(x) + F(x) \leq 3$.	Same inequality can be imposed, but $T(x)$ may take 1^+ (depending on convention).
Relationship	Baseline model without a special absolute-truth label.	Extends SNS by distinguishing 1 from 1^+ in the truth component.

Table 1: Comparison of single-valued neutrosophic sets and neutrosophic sets with an absolute-truth marker.

1.4 Belief Fuzzy Set

A Belief Fuzzy Set pairs a fuzzy membership function with a belief degree, expressing both graded membership and the credibility of that membership assessment.

Definition 1.5 (Belief Fuzzy Set: membership with credibility)**.** Let X be a universe. A *belief fuzzy set* (credibility form) is a pair (μ, β) where

$$\mu : X \to [0, 1], \qquad \beta : X \to [0, 1].$$

Here $\mu(x)$ is the membership degree and $\beta(x)$ is the credibility/belief in that assessment. A simple combined score is

$$\mu_{\text{eff}}(x) := \beta(x)\, \mu(x) \in [0, 1].$$

Definition 1.6 (Absolutely true element in a Belief Fuzzy Set)**.** Let (μ, β) be a belief fuzzy set on a universe X, where $\mu : X \to [0, 1]$ and $\beta : X \to [0, 1]$. An element $x \in X$ is said to be *absolutely true* in (μ, β) if

$$\mu(x) = 1 \qquad \text{and} \qquad \beta(x) = 1.$$

Equivalently, x is absolutely true iff it has maximal membership and maximal credibility simultaneously.

The comparison of Fuzzy Sets and Belief Fuzzy Sets is presented in Table 2.

2 Results

This section presents the main results of the paper.

Item	Fuzzy Set	Belief Fuzzy Set
Underlying data	A single membership map $\mu : X \to [0,1]$.	A pair (μ, β) with $\mu : X \to [0,1]$ and belief $\beta : X \to [0,1]$.
Meaning	Stores graded membership only.	Stores graded membership plus credibility (reliability) of that membership.
Per-element output	$\mu(x) \in [0,1]$.	$\big(\mu(x), \beta(x)\big) \in [0,1] \times [0,1]$.
Reduction relation	—	If $\beta(x) = 1$ for all x, then (μ, β) reduces to the fuzzy set μ.
One combined score (optional)	$\mu(x)$.	$\mu_{\text{eff}}(x) := \beta(x)\mu(x) \in [0,1]$.
Absolute truth (one natural choice)	x is (fully) true if $\mu(x) = 1$.	x is absolutely true if $\mu(x) = 1$ and $\beta(x) = 1$.

Table 2: Comparison of Fuzzy Sets and Belief Fuzzy Sets.

2.1 Belief Uncertain Set

A Belief Uncertain Set assigns each element a model-based uncertainty value and a belief degree, representing both the assessment and its reliability or credibility.

Definition 2.1 (Belief Uncertain Set (BU-Set))**.** Let X be a nonempty set and let M be an uncertainty model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$. A *Belief Uncertain Set of type* M on X is a pair

$$\mathbb{B} = (\mu_M, \beta)$$

consisting of two mappings

$$\mu_M : X \to \mathrm{Dom}(M), \qquad \beta : X \to [0,1].$$

For each $x \in X$, the value $\mu_M(x) \in \mathrm{Dom}(M)$ is the *M-uncertainty value* assigned to x, and $\beta(x) \in [0,1]$ is the *belief (credibility) degree* expressing how reliable the assessment $\mu_M(x)$ is regarded to be.

Remark 2.2 (Reduction to ordinary U-Sets)**.** If $\beta(x) = 1$ for all $x \in X$, then $\mathbb{B} = (\mu_M, \beta)$ carries no additional discounting and the Belief Uncertain Set reduces to the underlying U-Set $\mu_M : X \to \mathrm{Dom}(M)$.

Remark 2.3 (Optional belief-discounted uncertainty value)**.** Sometimes one wants to combine $\mu_M(x)$ and $\beta(x)$ into a single discounted uncertainty value. Write $\mu_M(x) = (u_1(x), \ldots, u_k(x)) \in [0,1]^k$ (coordinates in $[0,1]^k$). Define the coordinatewise discounting

$$\beta \odot \mu_M : X \to [0,1]^k, \qquad (\beta \odot \mu_M)(x) := \big(\beta(x)u_1(x), \ldots, \beta(x)u_k(x)\big).$$

If, in addition, $\mathrm{Dom}(M)$ is *star-shaped at the origin*, i.e.,

$$\forall a \in \mathrm{Dom}(M)\ \forall \lambda \in [0,1] \quad \lambda a \in \mathrm{Dom}(M),$$

then $(\beta \odot \mu_M)(x) \in \mathrm{Dom}(M)$ for all $x \in X$, so $\beta \odot \mu_M$ is again a U-Set of type M. (This extra closure property holds for many standard degree-domains such as $[0,1]^k$.)

Example 2.4 (A medical test assessment as a Belief Uncertain Set)**.** Let $X = \{p_1, p_2, p_3\}$ be a set of patients. Take the model M to be the single-valued neutrosophic model with degree-domain

$$\mathrm{Dom}(M) = [0,1]^3,$$

so that $\mu_M(x) = (T(x), I(x), F(x))$ records truth, indeterminacy, and falsity degrees.

Define $\mu_M : X \to [0,1]^3$ and $\beta : X \to [0,1]$ by

$$\mu_M(p_1) = (0.90,\ 0.05,\ 0.05), \qquad \beta(p_1) = 0.95,$$
$$\mu_M(p_2) = (0.55,\ 0.35,\ 0.10), \qquad \beta(p_2) = 0.60,$$
$$\mu_M(p_3) = (0.20,\ 0.20,\ 0.60), \qquad \beta(p_3) = 0.85.$$

Then $\mathbb{B} = (\mu_M, \beta)$ is a Belief Uncertain Set of type M on X. For instance, p_1 has high truth and high belief, reflecting a confident positive assessment, whereas p_2 has moderate truth but low belief, indicating that the available evidence is less reliable.

Theorem 2.5 (Belief Uncertain Sets generalize Belief Fuzzy Sets)**.** *Fix a universe X. Let M_{fz} be the uncertainty model whose degree-domain is*

$$\mathrm{Dom}(M_{\mathrm{fz}}) = [0,1] \subseteq [0,1]^1.$$

Then the following hold.

(i) For every Belief Fuzzy Set (μ, β) on X (i.e., $\mu : X \to [0,1]$ and $\beta : X \to [0,1]$), the pair

$$\mathbb{B} := (\mu_{M_{\mathrm{fz}}}, \beta) \quad \textit{with} \quad \mu_{M_{\mathrm{fz}}} := \mu$$

is a Belief Uncertain Set of type M_{fz}.

(ii) Conversely, every Belief Uncertain Set of type M_{fz} is (canonically) a Belief Fuzzy Set.

Hence, Belief Uncertain Sets strictly extend Belief Fuzzy Sets by allowing an arbitrary degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ in place of $[0,1]$.

Proof. (i) Let (μ, β) be a Belief Fuzzy Set on X. By definition, $\mu : X \to [0,1]$ and $\beta : X \to [0,1]$. Since $\mathrm{Dom}(M_{\mathrm{fz}}) = [0,1]$, the same map μ can be viewed as a map

$$\mu_{M_{\mathrm{fz}}} : X \to \mathrm{Dom}(M_{\mathrm{fz}}).$$

Therefore $(\mu_{M_{\mathrm{fz}}}, \beta)$ satisfies the type requirements in the definition of a Belief Uncertain Set of type M_{fz}, so it is a Belief Uncertain Set.

(ii) Let $\mathbb{B} = (\mu_{M_{\mathrm{fz}}}, \beta)$ be a Belief Uncertain Set of type M_{fz}. Then, by definition,

$$\mu_{M_{\mathrm{fz}}} : X \to \mathrm{Dom}(M_{\mathrm{fz}}) = [0,1], \qquad \beta : X \to [0,1].$$

Thus $(\mu_{M_{\mathrm{fz}}}, \beta)$ is exactly a Belief Fuzzy Set (credibility form).

Finally, for a general model M with $\mathrm{Dom}(M) \subseteq [0,1]^k$ one obtains a strictly broader class, because the first component may take values in higher-dimensional (or otherwise structured) degree-domains, while the fuzzy case corresponds to the special choice $k = 1$ and $\mathrm{Dom}(M) = [0,1]$. □

Definition 2.6 (Absolutely True element in a Belief Uncertain Set)**.** Let $\mathbb{B} = (\mu_M, \beta)$ be a Belief Uncertain Set of type M on a universe X, where $\mu_M : X \to \mathrm{Dom}(M) \subseteq [0,1]^k$ and $\beta : X \to [0,1]$.

Assume that the degree-domain $\mathrm{Dom}(M)$ has a distinguished *truth coordinate*

$$\pi_T : \mathrm{Dom}(M) \to [0,1],$$

so that $\pi_T(\mu_M(x))$ is interpreted as the *truth degree* of x under the model M.

An element $x \in X$ is said to be *absolutely true in* $\mathbb{B}$ if

$$\pi_T\big(\mu_M(x)\big) = 1 \qquad \text{and} \qquad \beta(x) = 1.$$

In this case, x has maximal truth degree and maximal belief (credibility) simultaneously.

Remark 2.7 (Neutrosophic instance)**.** If $M = M_{\mathrm{NS}}$ with $\mathrm{Dom}(M_{\mathrm{NS}}) = [0,1]^3$ and $\mu_M(x) = (T(x), I(x), F(x))$, then one may take $\pi_T(T, I, F) = T$. Hence x is absolutely true in $\mathbb{B}$ exactly when

$$T(x) = 1 \quad \text{and} \quad \beta(x) = 1.$$

The comparison of Uncertain Sets and Belief Uncertain Sets is given in Table 3. In addition, the related concept of MultiBelief Uncertain Sets is examined in Appendix A.

Item	**Uncertain Set (U-Set)**	**Belief Uncertain Set (BU-Set)**
Underlying data	A single assignment $\mu_M : X \to \mathrm{Dom}(M)$.	A pair (μ_M, β) with $\mu_M : X \to \mathrm{Dom}(M)$ and belief $\beta : X \to [0,1]$.
Meaning	Stores the model-dependent uncertainty value only.	Stores the uncertainty value plus credibility (reliability) of that value.
Outputs per element x	$\mu_M(x) \in \mathrm{Dom}(M)$.	$\big(\mu_M(x), \beta(x)\big) \in \mathrm{Dom}(M) \times [0,1]$.
Special case relation	—	If $\beta(x) = 1$ for all x, then BU-Set reduces to the U-Set μ_M.
Absolutely true notion (requires a truth coordinate)	If a truth projection $\pi_T : \mathrm{Dom}(M) \to [0,1]$ is fixed, one may call x true when $\pi_T(\mu_M(x)) = 1$.	With π_T fixed, one may call x absolutely true when $\pi_T(\mu_M(x)) = 1$ and $\beta(x) = 1$.

Table 3: Comparison of Uncertain Sets and Belief Uncertain Sets.

2.2 Uncertain set with an absolute-truth marker

An uncertain set with an absolute-truth marker assigns each element a model-dependent uncertainty value plus a special truth label distinguishing ordinary truth from absolute, universally valid truth.

Definition 2.8 (Extended unit scale with an absolute-truth marker)**.** Fix a formal symbol 1^+ and define the *marked unit scale*

$$[0,1]^+ := [0,1] \ \cup \ \{1^+\},$$

where $1^+ \notin [0,1]$ (it is not identified with any real number). We order $[0,1]^+$ by extending the usual order on $[0,1]$ and declaring

$$t < 1^+ \quad \text{for all } t \in [0,1].$$

Remark 2.9 (Crisp embedding and projection)**.** The inclusion (embedding) is the map

$$\iota : [0,1] \hookrightarrow [0,1]^+, \qquad \iota(t) = t.$$

A natural projection (forgetting the marker) is

$$\pi : [0,1]^+ \to [0,1], \qquad \pi(t) = t \ (t \in [0,1]), \quad \pi(1^+) = 1.$$

Definition 2.10 (Uncertain set with an absolute-truth marker (AT-U-Set))**.** Let X be a nonempty set and let M be an uncertainty model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$. An *Uncertain set with an absolute-truth marker of type M* on X is a mapping

$$\mu_M^+ : X \longrightarrow \mathrm{Dom}(M) \times [0,1]^+.$$

For each $x \in X$, write

$$\mu_M^+(x) = \big(u(x),\, t^+(x)\big), \qquad u(x) \in \mathrm{Dom}(M), \quad t^+(x) \in [0,1]^+.$$

We interpret $u(x)$ as the ordinary M-uncertainty value of x, and $t^+(x)$ as a *truth-marker degree* with the convention:

$$t^+(x) = 1^+ \implies x \text{ is } \textit{absolutely true}, \qquad t^+(x) = 1 \implies x \text{ is } \textit{(relatively) true}.$$

Remark 2.11 (Two useful special cases)**.** (1) *Pure marker form:* one may restrict the second component to $\{0, 1, 1^+\} \subset [0,1]^+$.

(2) *Compatibility with standard U-Sets:* setting $t^+(x) = \iota(1) = 1$ for all x recovers the underlying U-Set $x \mapsto u(x)$ while keeping a trivial marker layer.

Example 2.12 (A knowledge-base statement with an absolute-truth marker)**.** Let $X = \{s_1, s_2, s_3\}$ be a set of statements in a knowledge base, where

s_1 : "$2 + 2 = 4$", $\quad s_2$: "The device is currently online", $\quad s_3$: "The sensor reading is accurate".

Take the model M to be the fuzzy model with degree-domain $\mathrm{Dom}(M) = [0,1]$ (so $k = 1$). Define an AT-U-Set $\mu_M^+ : X \to [0,1] \times [0,1]^+$ by

$$\mu_M^+(s_1) = \big(1,\, 1^+\big), \qquad \mu_M^+(s_2) = \big(0.85,\, 1\big), \qquad \mu_M^+(s_3) = \big(0.70,\, 0.70\big).$$

Here the first component is the ordinary (fuzzy) truth assessment, while the second component is the truth-marker degree. Thus s_1 is declared *absolutely true* (marker 1^+), s_2 is treated as *relatively true* (marker 1) with high confidence 0.85 in the ordinary degree, and s_3 is neither relatively nor absolutely true since its marker value is below 1.

For reference, the comparison of Uncertain Sets and Uncertain Sets with an absolute-truth marker is provided in Table 4.

Item	**Uncertain Set (U-Set)**	**AT-U-Set (with absolute-truth marker)**
Underlying data	A single assignment $\mu_M : X \to \mathrm{Dom}(M)$.	A paired assignment $\mu_M^+ : X \to \mathrm{Dom}(M) \times [0,1]^+$.
Per-element output	$\mu_M(x) \in \mathrm{Dom}(M)$.	$\mu_M^+(x) = \big(u(x), t^+(x)\big)$ with $u(x) \in \mathrm{Dom}(M)$ and $t^+(x) \in [0,1]^+$.
Extra expressiveness	Encodes only the model-dependent uncertainty value.	Encodes the uncertainty value plus a truth-marker distinguishing 1 (relative truth) from 1^+ (absolute truth).
Absolute truth available?	Not natively; one can only speak about truth via the structure of $\mathrm{Dom}(M)$ (e.g., a truth coordinate).	Yes; by definition, $t^+(x) = 1^+$ declares x absolutely true (marker-based).
Relationship	—	If $t^+(x) = 1$ for all x, then μ_M^+ reduces to the underlying U-Set $x \mapsto u(x)$.
Forgetting the marker	Identity (nothing to forget).	Projection $\mathrm{pr}_1 : \mathrm{Dom}(M) \times [0,1]^+ \to \mathrm{Dom}(M)$ yields the underlying U-Set u.

Table 4: Comparison of Uncertain Sets and Uncertain Sets with an absolute-truth marker.

3 Conclusion

In this paper, we investigated mathematically coherent ways to represent both Truth and Absolutely Truth within degree-based and neutrosophic-style frameworks. As future work, we hope to see extensions of these ideas to structures such as graphs [19], hypergraphs [20], and superhypergraphs [14], together with computational experiments and concrete case studies that test and illustrate the proposed concepts in real-world settings.

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Data Availability

Since this research is purely theoretical and mathematical, no empirical data or computational analysis was utilized. Researchers are encouraged to expand upon these findings with data-oriented or experimental approaches in future studies.

Ethical Statement

As this study does not involve experiments with human participants or animals, no ethical approval was required.

Conflicts of Interest

The authors declare that they have no conflicts of interest related to the content or publication of this paper.

Code Availability

No code or software was developed for this study.

Clinical Trial

This study did not involve any clinical trials.

Consent to Participate

Not applicable.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Disclaimer

This work presents theoretical ideas and frameworks that have not yet been empirically validated. Readers are encouraged to explore practical applications and further refine these concepts. Although care has been taken to ensure accuracy and appropriate citations, any errors or oversights are unintentional. The perspectives and interpretations expressed herein are solely those of the authors and do not necessarily reflect the viewpoints of their affiliated institutions.

A Appendix: MultiBelief Uncertain Set

A MultiBelief Uncertain Set assigns each element an uncertainty value and several belief degrees, capturing multiple, possibly independent, credibility viewpoints.

Definition A.1 (MultiBelief Uncertain Set (MBU-Set))**.** Let X be a nonempty set and let M be an uncertainty model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$. Fix an integer $m \geq 1$. A *MultiBelief Uncertain Set of type M with m belief components* on X is a pair

$$\mathbb{MB} = (\mu_M, \boldsymbol{\beta})$$

where

$$\mu_M : X \to \mathrm{Dom}(M), \qquad \boldsymbol{\beta} : X \to [0,1]^m.$$

For each $x \in X$, the value $\mu_M(x)$ is the M-uncertainty value of x, and

$$\boldsymbol{\beta}(x) = \big(\beta_1(x), \ldots, \beta_m(x)\big) \in [0,1]^m$$

is the *multi-belief (multi-credibility) vector* attached to the assessment $\mu_M(x)$.

Remark A.2 (Canonical aggregation into a single belief degree)**.** Let $A : [0,1]^m \to [0,1]$ be any aggregation map (e.g., min, max, arithmetic mean, weighted mean). Given $\mathbb{MB} = (\mu_M, \boldsymbol{\beta})$, define the aggregated belief degree

$$\beta_A := A \circ \boldsymbol{\beta} : X \to [0,1], \qquad \beta_A(x) := A(\boldsymbol{\beta}(x)).$$

Then (μ_M, β_A) is a Belief Uncertain Set of type M.

Theorem A.3 (MultiBelief Uncertain Sets generalize Belief Uncertain Sets)**.** *Fix X and an uncertainty model M with $\mathrm{Dom}(M) \subseteq [0,1]^k$. Let $\mathrm{MBU}_m(M)$ denote the class of MultiBelief Uncertain Sets of type M with m belief components, and let $\mathrm{BU}(M)$ denote the class of Belief Uncertain Sets of type M.*

For each $m \geq 1$ there exists an injective mapping

$$\iota_m : \mathrm{BU}(M) \longrightarrow \mathrm{MBU}_m(M)$$

such that every Belief Uncertain Set is realized as a special case of a MultiBelief Uncertain Set. In particular, $\mathrm{MBU}_m(M)$ is a genuine generalization of $\mathrm{BU}(M)$ for every $m \geq 2$.

Proof. Fix $m \geq 1$. Let $\mathbb{B} = (\mu_M, \beta) \in \mathrm{BU}(M)$, so $\mu_M : X \to \mathrm{Dom}(M)$ and $\beta : X \to [0,1]$.

Define $\iota_m(\mathbb{B}) = (\mu_M, \boldsymbol{\beta})$ by keeping the same uncertainty assignment μ_M and setting, for each $x \in X$,

$$\boldsymbol{\beta}(x) := \big(\beta(x), \beta(x), \ldots, \beta(x)\big) \in [0,1]^m.$$

Then $\boldsymbol{\beta} : X \to [0,1]^m$ is well-defined, so $(\mu_M, \boldsymbol{\beta})$ is an element of $\mathrm{MBU}_m(M)$ by the definition of MultiBelief Uncertain Set.

It remains to show that ι_m is injective. Suppose $\iota_m(\mu_M, \beta) = \iota_m(\mu'_M, \beta')$. Then, by equality of ordered pairs, $\mu_M = \mu'_M$ and $\boldsymbol{\beta} = \boldsymbol{\beta}'$. For each $x \in X$, we have

$$(\beta(x), \ldots, \beta(x)) = \boldsymbol{\beta}(x) = \boldsymbol{\beta}'(x) = (\beta'(x), \ldots, \beta'(x)).$$

Comparing the first coordinates yields $\beta(x) = \beta'(x)$ for all x, hence $\beta = \beta'$. Therefore $(\mu_M, \beta) = (\mu'_M, \beta')$, proving injectivity.

Thus every Belief Uncertain Set embeds as the diagonal special case of an m-component MultiBelief Uncertain Set, so MultiBelief Uncertain Sets generalize Belief Uncertain Sets. For $m \geq 2$ the generalization is strict since $\mathrm{MBU}_m(M)$ also contains maps $\boldsymbol{\beta} : X \to [0,1]^m$ that are not coordinatewise constant. □

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